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Some Existence and Regularity Results for Dual Linear Control Problems

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1. INTRODUCTION

In this paper we will investigate dual linear control problems and situations where these will have optimal solutions where the states are absolutely continuous. These results are analogous to those obtained in the literature in Continuous Linear Programming [2, 5, 13], where although the primal and dual problems are formulated in such a way that an optimal solution may correspond to a measure, assumptions are made so that optimal solutions will exist where the control is, as usual, an L^1 function. Such difficulties may arise because for linear control problems the usual growth conditions encountered in theorems for the existence of an optimal solution are not satisfied. However, extensions of these theorems may be found in [7, 10], which enable us to determine whether optimal solutions do exist, but in a larger space. In line with the extended control problem formulation of [7], we define both the usual type of problem and an extended verson. We place assumptions on the extended version so that an optimal solution will exist in the extended sense and also so that there will be a solution in the usual sense. In the last section we also mention some regularity results for certain nonlinear control problems.

2. FORMULATION AND DUALITY

The notation we use is the same as in [7]: T is the fixed time interval $[t_0, t_1]$.

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$\mathscr{A}_m(D)$ is the space of vector valued functions $f=(f_1,...,f_m)$, where each f_i is absolutely continuous over the interval D.

$\mathscr{B}_m(D)$ is the space of vector valued functions $f=(f_1,...,f_m)$, where each f_i is of bounded variation on D.

$\mathscr{C}_m(D)$ is the space of vector valued functions $f=(f_1,...,f_m)$, where each f_i is continuous over the interval D.

$L^k_m(D)$ is the space of vector valued functions $f=(f_1,...,f_m)$, where each f_i belongs to $L^k(D)$.

$\mathscr{M}(D)$ is the space of one dimensional, nonnegative regular Borel measures on D.

When $D=T$ we will just write $\mathscr{A}_m$, $\mathscr{B}_m$, etc., and when $m=n$, the dimension of the state space, we will drop the subscript as well.

The $\|\cdot\|_V$ norm over the space $\mathscr{B}$ is defined as follows: let $x\in\mathscr{B}$, then $dx(t)=\dot{x}(t)\,dt+\xi(t)\,d\theta(t)$ for some $\theta\in\mathscr{M}$ and Borel measurable function ξ. We define

$$\|x\|_V=|x(t_0)|+\int_T|\dot{x}(t)|\,dt+\int_T|\xi(t)|\,d\theta(t).$$

This value is independent of the choice of θ and ξ.

A multifunction $S\colon T\rightrightarrows\mathbb{R}^n$ is upper semicontinuous if whenever K is a compact subset of $\mathbb{R}^n$, the set $\{t\in T{:}K\cap S(t)\neq\varnothing\}$ is closed.

A multifunction $S{:}T\rightrightarrows\mathbb{R}^n$ is lower semicontinuous if the set $\{t\in T{:}U\cap S(t)\neq\varnothing\}$ is open relative to T for every open $U\in\mathbb{R}^n$.

A multifuncton $S{:}T\rightrightarrows\mathbb{R}^n$ is fully lower semicontinuous if S is lower semicontinuous and one has $x_0\in\operatorname{cl}S(\tau)$ whenever there are neighbourhoods U and V of x_0 and τ such that the set $\{t\in V\colon S(t)\supset U\}$ is dense in V. (This definition is taken from [8, p. 457].)

Consider the following linear optimal control problem which we shall label (P) for primal:

(P) minimize $\tilde{\Phi}(x,u)=l(x(t_0),x(t_1))+\int_T[a(t)\,x(t)+b(t)\,u(t)]\,dt$ subject to

$$\dot{x}(t)=A(t)\,x(t)+B(t)\,u(t)+c(t) \tag{1}$$

$$C(t)\,x(t)+D(t)\,u(t)+d(t)\geqslant 0 \tag{2}$$

$$u(t)\geqslant 0 \tag{3}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and for each $t\in T$, $A(t)\in\mathbb{R}^{n\times m}$, $B(t)\in\mathbb{R}^{n\times m}$, $C(t)\in\mathbb{R}^{k\times n}$, $D(t)\in\mathbb{R}^{k\times m}$, $c(t)\in\mathbb{R}^n$, $d(t)\in\mathbb{R}^k$, $a^*(t)\in\mathbb{R}^n$ and $b^*(t)\in\mathbb{R}^m$ (* denotes the transpose) with the components of the above being L^∞_1 functions of t, b and d Borel measurable, $x\in\mathscr{A}$, $u\in L^1_m$, and $l(\cdot,\cdot)$ a lower semicontinuous function on $\mathbb{R}^n\times\mathbb{R}^n$.

Unless the constraints (2) and (3) are such that for each x the components of u are bounded, then problem (P) will not satisfy the usual growth conditions needed to ensure the existence of an optimal solution with $x \in \mathscr{A}$. This is brought about by the linearity of the problem in u.

What can happen is that a minimizing sequence will lead to a discontinuous state vector x, where the discontinuity is caused by an impulse control, and so an optimal solution will not exist. To overcome this we extend the problem so that it not only covers states $x \in \mathscr{A}$ but those that belong to $\mathscr{B}$ as well, and therefore we most include extended controls which can bring about, amongst other things, the jumps in the state vector.

Such extensions and related existence results for optimal solutions can be found in [7, 10, 12]. Hence, let us define the extended primal problem (EP).

(EP) minimize $\Phi(x, v) = l(x(t_0), x(t_1)) + \int_T [a(t)\, x(t) + b(t)\, u(t)]\, dt + \int_T b(t)\, \mu(t)\, d\theta(t)$ subject to $x \in \mathscr{B}$, $v \in \mathscr{B}_m$, $\theta \in \mathscr{M}$ and

$$dv(t) = u(t)\, dt + \mu(t)\, d\theta(t) \tag{4}$$

$$dx(t) = [A(t)\, x(t) + c(t)]\, dt + B(t)\, dv(t) \tag{5}$$

$$C(t)\, x(t) + D(t)\, u(t) + d(t) \geqslant 0 \tag{6}$$

$$D(t)\, \mu(t) \geqslant 0 \tag{7}$$

$$u(t) \geqslant 0 \tag{8}$$

$$\mu(t) \geqslant 0. \tag{9}$$

Due to the linearity of the expressions involved in (P) we can formulate a dual problem which we shall call (D).

(D) minimize $\tilde{\Psi}(p, w) = m(p(t_0), p(t_1)) + \int_T [p(t)\, c(t) + w(t)\, d(t)]\, dt$ subject to

$$\dot{p}(t) = -p(t)\, A(t) - w(t)\, C(t) + a(t) \tag{10}$$

$$p(t)\, B(t) + w(t)\, D(t) - b(t) \leqslant 0 \tag{11}$$

$$w(t) \geqslant 0 \tag{12}$$

where $p^* \in \mathbb{R}^n$, $w^* \in \mathbb{R}^k$ and

$$m(p_0, p_1) = \sup_{x_0, x_1} \{p_0 x_0 - p_1 x_1 - l(x_0, x_1)\}. \tag{13}$$

As for the primal problem, we can define an extended dual problem,

(ED) minimize $\Psi(p, v) = m(p(t_0), p(t_1)) + \int_T [p(t)\, c(t) + w(t)\, d(t)]\, dt + \int_T \omega(t)\, d(t)\, d\tilde{\theta}(t)$ subject to $p^* \in \mathscr{B}_n$, $v^* \in \mathscr{B}_k$,

$$dv(t) = w(t)\, dt + \omega(t)\, d\tilde{\theta}(t) \tag{14}$$

$$dp(t) = [-p(t)\, A(t) + a(t)]\, dt - dv(t)\, C(t) \tag{15}$$

$$p(t)\,B(t)+w(t)\,D(t)-b(t)\leqslant 0 \tag{16}$$

$$\omega(t)\,D(t)\leqslant 0 \tag{17}$$

$$w(t)\geqslant 0 \tag{18}$$

$$\omega(t)\geqslant 0 \tag{19}$$

and $\tilde{\theta}\in\mathcal{M}$.

Note that since b and d are Borel measurable, the integrals Φ and Ψ will be well defined although possibly infinite.

DEFINITION.

$$X(t)=\{x: \exists u\geqslant 0 \text{ such that } C(t)\,x+D(t)\,u+d(t)\geqslant 0\}$$

$$\bar{P}(t)=\{p: \exists w\geqslant 0 \text{ such that } pB(t)+wD(t)-b(t)\leqslant 0\}.$$

Since $X(t)$ and $\bar{P}(t)$ are the projections of closed, polyhedral convex sets for each t, they are themselves closed and convex.

Following the guidance of linear programming, we will now prove a weak duality result for (EP) and (ED).

THEOREM 1. *Assume X and $\bar{P}$ are upper semicontinuous multifunctions on T. Then*

$$\Phi(x, v)\geqslant -\Psi(p, \upsilon) \tag{20}$$

for all feasible (x, v) and (p, υ).

Proof. Suppose (x, v) and (p, υ) are feasible for their respective problems. Then

$$\begin{aligned}\Phi(x, v)&=l(x(t_0), x(t_1))+\int_T [a(t)\,x(t)+b(t)\,u(t)]\,dt\\&\quad+\int_T b(t)\,\mu(t)\,d\theta(t)\\&\geqslant p(t_0)\,x(t_0)-p(t_1)\,x(t_1)-m(p(t_0), p(t_1))\\&\quad+\int_T [a(t)\,x(t)+b(t)\,u(t)]\,dt\\&\quad+\int_T b(t)\,\mu(t)\,d\theta(t)\qquad(\text{using (13)})\end{aligned}$$

$$\geqslant p(t_0)\,x(t_0) - p(t_1)\,x(t_1) - m(p(t_0), p(t_1))$$
$$+ \int_T [a(t)\,x(t) + p(t)\,B(t)\,u(t) + w(t)\,D(t)\,u(t)]\,dt$$
$$+ \int_T \sup_{\substack{\tilde{p} \in \bar{P}(t) \\ \tilde{w} \geqslant 0}} \{\tilde{p}B(t)\,\mu(t) + \tilde{w}D(t)\,\mu(t)\}\,d\theta(t)$$
(using (8), (9) and (16))
$$\geqslant p(t_0)\,x(t_0) - p(t_1)\,x(t_1) - m(p(t_0), p(t_1))$$
$$+ \int_T [a(t)\,x(t) + p(t)\,B(t)\,u(t) - w(t)\,C(t)\,x(t) - w(t)\,d(t)]\,dt$$
$$+ \int_T \sup_{\substack{\tilde{p} \in \bar{P}(t) \\ \tilde{w} \geqslant 0}} \{\tilde{p}B(t)\,\mu(t) + \tilde{w}D(t)\,\mu(t)\}\,d\theta(t)$$
(using (18), (19) and (6))
$$\geqslant p(t_0)\,x(t_0) - p(t_1)\,x(t_1) - m(p(t_0), p(t_1))$$
$$+ \int_T [a(t)\,x(t) + p(t)\,B(t)\,u(t) - w(t)\,C(t)\,x(t) - w(t)\,d(t)]\,dt$$
$$+ \int_T \sup_{\tilde{p} \in \bar{P}(t)} \{\tilde{p}B(t)\,\mu(t)\}\,d\theta(t)$$
$$+ \int_T \sup_{\tilde{u} \geqslant 0} \{\omega(t)\,D(t)\,\tilde{u}\}\,d\tilde{\theta}(t) \qquad \text{(using (7), (8), (17) and (18))}$$
$$\geqslant p(t_0)\,x(t_0) - p(t_1)\,x(t_1) - m(p(t_0), p(t_1))$$
$$+ \int_T [a(t)\,x(t) + p(t)\,B(t)\,u(t) - w(t)\,C(t)\,x(t) - w(t)\,d(t)]\,dt$$
$$+ \int_T \sup_{\tilde{p} \in \bar{P}(t)} \{\tilde{p}B(t)\,\mu(t)\}\,d\theta(t) - \int_T \omega(t)\,d(t)\,d\tilde{\theta}(t)$$
$$+ \int_T \sup_{\tilde{x} \in X(t)} \{-\omega(t)\,C(t)\,\tilde{x}\}\,d\tilde{\theta}(t) \qquad \text{(using (6) and (19))}$$
$$= -\Psi(p, v) + p(t_0)\,x(t_0) - p(t_1)\,x(t_1) \tag{21}$$
$$+ \int_T [\dot{p}(t)\,x(t) + p(t)\,\dot{x}(t)]\,dt$$
$$+ \int_T \sup_{\tilde{x} \in X(t)} \{-\omega(t)\,C(t)\,\tilde{x}\}\,d\tilde{\theta}(t)$$
$$+ \int_T \sup_{\tilde{p} \in \bar{P}(t)} \{\tilde{p}B(t)\,\mu(t)\}\,d\theta(t) \qquad \text{(using (5) and (15)).}$$

Let $x_+ \in \mathscr{B}$ be right continuous, $p_- \in \mathscr{B}$ left continuous, and $x_+ = x$, $p_- = p$ almost everywhere. Then by Proposition 1 of [10],

$$\begin{aligned} p(t_1)\,x(t_1) - p(t_0)\,x(t_0) = \int_T & [p(t)\,\dot{x}(t) + \dot{p}(t)\,x(t)]\,dt \\ & + \int_{(t_0,t_1)} \frac{dp}{d\tilde{\theta}}(t)\,x_+(t)\,d\tilde{\theta}(t) \\ & + \int_{(t_0,t_1)} p_-(t)\,\frac{dx}{d\theta}(t)\,d\theta(t) \\ & + \Delta p(t_0)\,x_+(t_0) + p(t_0)\,\Delta x(t_0) \\ & + \Delta p(t_1)\,x(t_1) + p_-(t_1)\,\Delta x(t_1). \end{aligned} \tag{22}$$

Since X and $\bar{P}$ are upper semicontinuous multifunctions (21) and (22) imply

$$\Phi(x, v) \geqslant -\Psi(p, \upsilon). \qquad \text{Q.E.D.}$$

COROLLARY 1. *Let X and $\bar{P}$ be upper semicontinuous multifunctions. Then if (x, v) and (p, υ) are feasible for their respective problems and*

$$\Phi(x, v) = -\Psi(p, \upsilon) \tag{23}$$

we have that (x, v) and (p, υ) are optimal.

Remark. Since the problems (P) and (D) are contained in their extended problems, analogous versions of (20) and (23) hold, but for (P) and (D) we do not need the upper semicontinuity assumptions.

The next theorem gives sufficient conditions for optimality and is modelled on the complementary slackness principle of linear programming.

THEOREM 2. *Let X and $\bar{P}$ be upper semicontinuous multifunctions. If (x, v) and (p, υ) are feasible and the following conditions hold*

$$m(p(t_0), p(t_1)) = p(t_0)\,x(t_0) - p(t_1)\,x(t_1) - l(x(t_0), x(t_1)) \tag{24}$$

$$\begin{aligned} [p(t)\,B(t) + w(t)\,D(t) - b(t)]_i < 0 &\Rightarrow u_i(t) = 0 \\ &\Rightarrow \mu_i(t) = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} [C(t)\,x(t) + D(t)\,u(t) + d(t)]_j > 0 &\Rightarrow w_j(t) = 0 \\ &\Rightarrow \omega_j(t) = 0 \end{aligned} \tag{26}$$

$$[\omega(t)\,D(t)]_i < 0 \Rightarrow u_i(t) = 0 \tag{27}$$

$$[D(t)\,\mu(t)]_j > 0 \Rightarrow w_j(t) = 0 \tag{28}$$

$$B(t)\,\mu(t) \text{ is normal to } \bar{P}(t) \text{ at } p_+(t) \text{ and } p_-(t) \tag{29}$$

$$-\omega(t)\,C(t) \text{ is normal to} X(t) \text{ at } x_+(t) \text{ and } x_-(t) \tag{30}$$

then (x, v) *and* (p, υ) *are optimal for their problems.*

Proof. If we follow the steps in the proof of Theorem 1, we see that we can replace the inequalities with equalities if we use the above conditions. Conditions (29) and (30) allow us to substitute

$$p_-(t)\frac{dx}{d\theta}(t) \text{ for } \sup_{\tilde{p} \in \bar{P}(t)} \{\tilde{p}B(t)\,\mu(t)\}$$

and

$$\frac{dp}{d\theta}(t)\,x_+(t) \text{ for } \sup_{\tilde{x} \in X(t)} \{-\omega(t)\,C(t)\,\tilde{x}\}.$$

We thus obtain

$$\Phi(x, v) = -\Psi(p, \upsilon)$$

and by Corollary 1, (x, v) and (p, υ) are optimal. Q.E.D.

Now we wish to determine conditions under which the optimal solutions to (EP) and (ED) will also be solutions to the original problems (P) and (D). This will imply $x, p \in \mathscr{A}$.

THEOREM 3. *If, for all* $t \in T$, *the following is true*

$$\{\mu : D(t)\,\mu \geqslant 0, \mu \geqslant 0\} = \{0\} \tag{31}$$

then any feasible (x, v) *for* (EP) *must have* $\mu(t) \equiv 0$. *So any optimal solution* $(\tilde{x}, \tilde{v})$ *of* (EP) *is trivially equivalent to an optimal pair* $(\bar{x}, \bar{u})$ *with* $\bar{x} \in \mathscr{A}$, $\bar{u} \in L^1_m$ *and* $(\bar{x}, \bar{u})$ *optimal for* (P).

For completeness we will state the dual version of Theorem 3.

THEOREM 4. *If, for all* $t \in T$, *the following is true*

$$\{\omega : \omega D(t) \leqslant 0, \omega \geqslant 0\} = \{0\} \tag{32}$$

then any feasible (p, υ) *for* (ED) *must have* $\omega(t) \equiv 0$. *So any optimal solution* $(\tilde{p}, \tilde{\upsilon})$ *of* (ED) *is trivially equivalent to an optimal pair* $(\bar{p}, \bar{w})$ *with* $\bar{p} \in \mathscr{A}$, $\bar{w} \in L^1_k$ *and* $(\bar{p}, \bar{w})$ *optimal for* (D).

The next two theorems display other conditions under which an optimal solution for the extended problem will also be optimal for the original problem; in other words, the state vector $\bar{x}$ will correspond to an absolutely continuous function.

THEOREM 5. *Let* $(\bar{x}, \bar{v})$ *be optimal for* (EP), *where*

$$d\bar{v}(t) = \bar{u}(t)\,dt + \bar{\mu}(t)\,d\bar{\theta}(t).$$

Assume any of the following:

(i) *For all* $t \in T$, $A(t) \geqslant 0$, $B(t) \leqslant 0$, $C(t) \geqslant 0$, $a(t) \leqslant 0$ *and* $b(t) \geqslant 0$. *Also* $l(x(t_0), x(t_1)) = \bar{l}(x(t_0))$.

(ii) *For all* $t \in T$, $A(t) \geqslant 0$, $B(t) \geqslant 0$, $C(t) \leqslant 0$, $a(t) \geqslant 0$ *and* $b(t) \geqslant 0$. *Also* $l(x(t_0), x(t_1)) = \bar{l}(x(t_0))$.

(iii) *For all* $t \in T$, $A(t) \leqslant 0$, $B(t) \leqslant 0$, $C(t) \leqslant 0$, $a(t) \geqslant 0$ *and* $b(t) \geqslant 0$. *Also* $l(x(t_0), x(t_1)) = \hat{l}(x(t_1))$.

(iv) *For all* $t \in T$, $A(t) \leqslant 0$, $B(t) \geqslant 0$, $C(t) \geqslant 0$, $a(t) \leqslant 0$ *and* $b(t) \geqslant 0$. *Also* $l(x(t_0), x(t_1)) = \hat{l}(x(t_1))$.

Then there exists an optimal pair $(\tilde{x}, \tilde{v})$ *for* (EP), *where*

$$d\tilde{v}(t) = \bar{u}(t)\,dt \qquad \text{and} \qquad \tilde{x} \in \mathscr{A}.$$

Proof. *Case* (i). Let $\tilde{x}(t_0) = \bar{x}(t_0)$ so that $\bar{l}(\tilde{x}(t_0)) = \bar{l}(\bar{x}(t_0))$. Since $\bar{\mu}(t) \geqslant 0$ and $B(t) \leqslant 0$ it follows that

$$\begin{aligned} d\bar{x}(t) &= [A(t)\,\bar{x}(t) + B(t)\,\bar{u}(t) + c(t)]\,dt + B(t)\,\bar{\mu}(t)\,d\bar{\theta}(t) \\ &\leqslant [A(t)\,\bar{x}(t) + B(t)\,\bar{u}(t) + c(t)]\,dt \end{aligned}$$

and at t_0 the right-hand side equals $d\tilde{x}(t)$.

Since $A(t) \geqslant 0$ we then have $\bar{x}(t) \leqslant \tilde{x}(t)$ for all $t \in T$. Then

$$\begin{aligned} &\int_T [a(t)\,\bar{x}(t) + b(t)\,\bar{u}(t)]\,dt + \int_T b(t)\,\bar{\mu}(t)\,d\bar{\theta}(t) \\ &\qquad \geqslant \int_T [a(t)\,\tilde{x}(t) + b(t)\,\bar{u}(t)]\,dt. \end{aligned}$$

Hence $\Phi(\bar{x}, \bar{v}) \geqslant \Phi(\tilde{x}, \tilde{v})$. We still need to verify the feasibility of $(\tilde{x}, \tilde{v})$. In light of $\bar{v}$ being feasible for $\bar{x}$ this only requires the following:

$$C(t)\,\tilde{x}(t) \geqslant C(t)\,\bar{x}(t) \geqslant -d(t) - D(t)\,\bar{u}(t).$$

Hence $(\tilde{x}, \tilde{v})$ is optimal.

The proofs for the other cases are minor variations of the above. Q.E.D.

COROLLARY 2. *Assume* (32) *and any of the cases of Theorem* 5. *Then if optimal solutions* (x, v) *and* (p, υ) *exist for* (EP) *and* (ED), *the states* x *and* p *both belong to* $\mathscr{A}$.

One might wonder why we do not combine conditions (31) and (32) to obtain a similar result. Using Tucker's theorem of the alternative [6, p. 29] we see that if (31) holds then $\exists\omega$ that violates (32) and vice versa. That is, (31) and (32) are mutually exclusive.

The dual version of Theorem 5 is the following.

THEOREM 6. *Let* $(\bar{p}, \bar{\upsilon})$ *be optimal for* (ED), *where*

$$d\bar{\upsilon}(t) = \bar{w}(t)\, dt + \bar{\omega}(t)\, d\bar{\theta}(t).$$

Assume any of the following:

(i) *For all* $t \in T$, $A(t) \leqslant 0$, $B(t) \leqslant 0$, $C(t) \geqslant 0$, $c(t) \leqslant 0$ *and* $d(t) \geqslant 0$. *Also* $m(p(t_0), p(t_1)) = \tilde{m}(p(t_0))$.

(ii) *For all* $t \in T$, $A(t) \leqslant 0$, $B(t) \geqslant 0$, $C(t) \leqslant 0$, $c(t) \geqslant 0$ *and* $d(t) \geqslant 0$. *Also* $m(p(t_0), p(t_1)) = \tilde{m}(p(t_0))$.

(iii) *For all* $t \in T$, $A(t) \geqslant 0$, $B(t) \geqslant 0$, $C(t) \geqslant 0$, $c(t) \geqslant 0$ *and* $d(t) \geqslant 0$. *Also* $m(p(t_0), p(t_1)) = \hat{m}(p(t_1))$.

(iv) *For all* $t \in T$, $A(t) \geqslant 0$, $B(t) \leqslant 0$, $C(t) \leqslant 0$, $c(t) \leqslant 0$ *and* $d(t) \geqslant 0$. *Also* $m(p(t_0), p(t_1)) = \hat{m}(p(t_1))$.

Then there exists an optimal pair $(\tilde{p}, \tilde{\upsilon})$ *for* (ED), *where* $d\tilde{\upsilon}(t) = \bar{w}(t)\, dt$ *and* $\tilde{p} \in \mathscr{A}$.

COROLLARY 3. *Assume* (31) *and any of the cases of Theorem* 6 *hold. Then if optimal solutions* (x, v) *and* (p, υ) *exist for* (EP) *and* (ED), *the states* x *and* p *both belong to* $\mathscr{A}$.

3. EXISTENCE AND REGULARITY

Having determined conditions under which any optimal solution to (EP) or (ED) will be absolutely continuous instead of being merely of bounded variation, we now wish to use the results of [7] to ascertain when an optimal solution will actually exist for the various cases outlined in the theorems. To do this we will show that the assumptions of [7] hold. In the terminology of that paper, these are:

(General). The Lagrangian L is a Lebesgue normal integrand, $L(t, x, \cdot)$ is convex for each (t, x) and l is lower semicontinuous.

(A2) $\bar{P}$ is fully lower semicontinuous on T and int $\bar{P}(t) \neq \emptyset$ for any $t \in T$.

(A3) X is upper semicontinuous and $X(t)$ is closed for each $t \in T$.

(A4) For each $x \in \mathscr{B}$ such that $x(t) \in X(t)$ a.e. one has

$$\int_V |H(t, x(t), p)| \, dt < \infty$$

whenever V is an open subset of T and p is a point of $\mathbb{R}^n$ having a neighbourhood U such that $U \subset \bar{P}(t)$, $\forall t \in V$.

(A5) For each $M \geqslant 0$ and function $p \in \mathscr{E}$, where

$$\mathscr{E} = \{ p \in \mathscr{C} : p(t) \in \operatorname{int} \bar{P}(t), \forall t \in T \}$$

there exists an integrable function $\Gamma: T \to \mathbb{R}$ such that whenever $x \in \mathscr{B}$ $x(t) \in X(t)$ a.e. and $\| x \|_V \leqslant M$, then

$$H(t, x(t), p(t)) \leqslant \Gamma(t) \qquad \text{a.e.}$$

(U1) For each fixed $t \in T$, $\alpha \in \mathbb{R}$ and bounded set $S \subset \mathbb{R}^n \times \mathbb{R}^n$, the set

$$\{ u \in \mathbb{R}^m : \exists (x, v) \in S \text{ with } K(t, x, v, u) \leqslant \alpha \}$$

is bounded.

Note that the following would imply both (A4) and (A5)

For each $M \geqslant 0$ there exists a summable function $\Gamma: T \to \mathbb{R}$ such that

$$| H(t, x, p) | \leqslant \Gamma(t)$$

when $x \in X(t)$, $p \in \bar{P}(t)$, $|x| \leqslant M$ and $|p| \leqslant M$.

If we can place appropriate conditions on (EP) to ensure that the above hold then we can use Theorem 4A of [7] to determine that an optimal state vector $x \in \mathscr{B}$ will exist. By combining this with Theorem 5 we will then have an optimal $x \in \mathscr{A}$ so that (P) also will have an optimal solution.

Rather than going through all the cases, we shall concentrate on case (i) of Theorem 5. The same methods with the appropriate changes in the assumptions will produce similar results for the other cases.

We can convert (P) to a problem with the same form as in [7] by the following method. Define K by

$$\begin{aligned} K(t, x, v, u) &= a(t)\, x + b(t)\, u \qquad && \text{if } v = A(t)\, x + B(t)\, u + c(t), \\ & && C(t)\, x + D(t)\, u + d(t) \geqslant 0,\ u \geqslant 0 \\ &= +\infty && \text{otherwise.} \end{aligned}$$

Using Lemma 6 of [9] we see that K is a normal integrand. The Lagrangian for both (P) and (EP) is

$$L(t, x, v) = \inf_u K(t, x, v, u).$$

The following are the assumptions we will need to make to show that (A2)–(A5) and (U1) are satisfied:

(Q1) the components of b and B are continuous functions;

(Q2) for each $t \in T$ the components of $D(t)$ are nonnegative and there exists an L_1^1 function g such that $g(t) > 0$ for all $t \in T$, and for each t and row i of $D(t)$, there exists a j such that $d_{ij}(t) \geqslant 1/g(t)$;

(Q3) there exists a $\delta > 0$ such that $b_i(t) \geqslant \delta$ for all i and all $t \in T$.

For convenience we label the assumptions in case (i) of Theorem 5 as (Q0). It is easy to see that the following is true.

PROPOSITION 1. *If* (Q3) *holds then so does* (U1).

PROPOSITION 2. *If* (Q2) *holds then so does* (A3). *Moreover,* $X(t) \equiv \mathbb{R}^n$.

Proof.

$$X(t) = \{x : \exists u \geqslant 0 \text{ with } C(t)\, x + D(t)\, u + d(t) \geqslant 0\}.$$

Choose an $x \in \mathbb{R}^n$. Suppose each component of $-C(t)\, x - d(t)$ is less than or equal to $M \in \mathbb{R}$; then by choosing $\bar{u}_i(t) \geqslant Mg(t)$ we will have each component of $D(t)\, \bar{u}(t)$ equal or greater than M. This means that for each $x \in \mathbb{R}^n$ we can find a $u \geqslant 0$ that will satisfy $C(t)\, x + D(t)\, u + d(t) \geqslant 0$ so $X(t) \equiv \mathbb{R}^n$ and (A3) is satisfied. Q.E.D.

Combining the conditions of the above propositions will allow us to reap the benefits of the Equivalence Theorem of [12], one of which is the normality of L, thereby verifying the (General) assumption. Also, if we can determine that an optimal solution exists for the associated calculus of variations problem, then we will know that an optimal solution will exist for (EP), and what is more, under the conditions of case (i), an optimal solution will exist for (P).

The following propositions will show that (A2), (A4) and (A5) hold.

PROPOSITION 3. *Let* (Q0), (Q1), (Q2) *and* (Q3) *be satisfied. Then* (32) *holds and* $\bar{P}$ *is continuous with nonempty interior. Hence* (A2) *holds.*

Proof. Condition (32) obviously holds so we only need concern ourselves with the assertions about $\bar{P}$. Due to the assumptions on $D(t)$ we can write $\bar{P}(t)$ as

$$\bar{P}(t) = \{p : pB(t) \leqslant b(t)\}.$$

Because of the signs of the components of $b(t)$ and $B(t)$, $\bar{P}(t)$ contains the nonnegative orthant and int $\bar{P}(t) \neq \emptyset$. For a multifunction to be continuous it must be both upper and lower semicontinuous. For $\bar{P}$ to be lower semicontinuous we must have for every open set $U = \mathbb{R}^n$ the set $V = \{t : U \cap \bar{P}(t) \neq \emptyset\}$ is relatively open in T. Fix an open set $U \subset \mathbb{R}^n$ and choose a $\tau \in V$. Since $\bar{P}(t)$ has nonempty interior there exists a $p_0 \in U \cap$ int $\bar{P}(\tau)$. Assumption (Q3) then implies

$$p_0 B(\tau) < b(\tau) - \varepsilon e \qquad \text{for some } \varepsilon > 0 \text{ (} e \text{ is the unit vector).}$$

To prove lower semicontinuity we need to show that there exists a $\delta > 0$ such that $p_0 \in \bar{P}(t)$ when $|t - \tau| < \delta$. Choose a $\delta > 0$ such that when $|t - \tau| < \delta$ we have

$$|B_{ij}(t) - B_{ij}(\tau)| < \frac{\varepsilon}{2n |p_0|} \qquad \forall i, j$$

$$|b_i(t) - b_i(\tau)| < \frac{\varepsilon}{2} \qquad \forall i.$$

Then

$$\begin{aligned} p_0 B(t) &= p_0 B(\tau) + p_0(B(t) - B(\tau)) \\ &< p_0 B(\tau) + \frac{\varepsilon}{2} e < b(\tau) - \frac{\varepsilon}{2} e < b(t). \end{aligned}$$

So $p_0 \in \bar{P}(t)$ when $|t - \tau| < \delta$. Therefore $\bar{P}$ is lower semicontinuous.

To prove $\bar{P}$ is upper semicontinuous we must show that for each compact set $K \subset \mathbb{R}^n$, the set $G = \{t : K \cap \bar{P}(t) \neq \emptyset\}$ is relatively closed in T, and since $\bar{P}$ is closed, this is equivalent to showing that, if $t_n \to \bar{t}$, $p_n \in K \cap \bar{P}(t_n)$ and $p_n \to \bar{p}$ then $\bar{p} \in K \cap \bar{P}(\bar{t})$. Obviously $\bar{p} \in K$. But also

$$p_n B(t_n) \leqslant b(t_n)$$

and using the continuity of B and b we have

$$\bar{p} B(\bar{t}) \leqslant b(\bar{t}).$$

So $\bar{p} \in \bar{P}(\bar{t})$ as well. Therefore $\bar{P}$ is lower semicontinuous and hence continuous. Q.E.D.

PROPOSITION 4. *Let* (Q2) *be satisfied. Then assumptions* (A4) *and* (A5) *hold for problem* (EP).

Proof. The Hamiltonian for (P) can be expressed as

$$H(t, x, p) = pA(t)\,x + pc(t) - a(t)\,x$$
$$+ \inf_{w} \{wC(t)\,x + wd(t)\colon pB(t) + wD(t) - b(t) \leqslant 0,\ w \geqslant 0\}.$$

The assumptions on D imply that for any $p \in \bar{P}(t)$, $w = 0$ is an admissible element in the above set. Hence

$$H(t, x, p) \leqslant pA(t)\,x + pc(t) - a(t)\,x \qquad \text{when } p \in \bar{P}(t). \tag{33}$$

An alternate expression for H is the following:

$$H(t, x, p) = pA(t)\,x + pc(t) - a(t)\,x$$
$$+ \sup_{u} \{\,pB(t)\,u - b(t)\,u\colon C(t)\,x + D(t)\,u + d(t) \geqslant 0,\ u \geqslant 0\}.$$

Suppose $|x| \leqslant M$, $\|d_i\|_\infty \leqslant \alpha$ and $\|C_{ij}\|_\infty \leqslant \beta$ for all i and j. Let $\bar{u}$ be chosen as follows:

$$\bar{u}_i(t) = (\alpha + \beta n M)\,g(t) \qquad \text{for each } i.$$

Then

$$D(t)\,\bar{u}(t) \geqslant -d(t) - C(t)\,x.$$

Hence

$$H(t, x, p) \geqslant pA(t)\,x + pc(t) - a(t)\,x$$
$$+ (\,pB(t) - b(t))(\alpha + \beta n M)\,g(t). \tag{34}$$

Referring to the comment after (A5), and combining the inequalities (33) and (34), we see that (A4) and (A5) are satisfied. Q.E.D.

We are now at the stage of being able to apply Theorem 1A of [7]. If we can place assumptions on the problem to ensure the boundedness of the level sets of Φ in the $\|\cdot\|_V$ norm, and if a feasible solution exists, then we will know that an optimal solution exists for (EP).

We will need the following assumption on the function $\bar{l}$:

(Q4) $\bar{l}$ is not identically $+\infty$ and there exists a function $k\colon [0, \infty] \to \mathbb{R} \cup \{+\infty\}$ which is nondecreasing and has the properties

$$\bar{l}(a) \geqslant k(|a|) \text{ with } \lim_{s \to +\infty} k(s)/s = +\infty.$$

Proposition 5. *Let* (Q0), (Q3) *and* (Q4) *be satisfied. Then the level set* $\{x \in \mathscr{B} : \Phi(x, v) \leqslant \alpha$ *for some* $v\}$ *is bounded in the* $\|\cdot\|_V$ *norm for any* $\alpha \in \mathbb{R}$.

Proof. Because of the signs of the components of B we must have

$$dx(t) \leqslant \dot{x}(t) \leqslant A(t)\, x(t) + c(t). \tag{35}$$

Let

$$\dot{y}(t) = A(t)\, y(t) + c(t)$$

then $y(t) = \Phi(t)\, \Phi^{-1}(t_0)\, y(t_0) + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, ds$ where Φ is a fundamental matrix for $\dot{y}(t) = A(t)\, y(t)$.

Let $x \in \mathscr{B}$ satisfy (35) and choose $y(t_0) = x(t_0)$. Since $A(t) \geqslant 0$ we have

$$x(t) \leqslant y(t) = \Phi(t)\, \Phi^{-1}(t_0)\, x(t_0) + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, ds.$$

Therefore if x is such that $\Phi(x, v) \leqslant \alpha$ we will have

$$\begin{aligned}
\alpha &\geqslant \bar{l}(x(t_0)) + \int_T a(t)\, x(t)\, dt \\
&\geqslant k(|x(t_0)|) + \int_T a(t)\, \Phi(t) \left[\Phi^{-1}(t_0)\, x(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, ds \right] dt \\
&\geqslant k(|x(t_0)|) + |x(t_0)| \int_T a(t) \ulcorner \Phi(t)\, \Phi^{-1}(t_0)\, e \urcorner\, dt \\
&\quad + \int_T a(t)\, \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, ds\, dt
\end{aligned}$$

where e is a vector whose components are unity and $\ulcorner K \urcorner$ for a matrix K makes each of its components nonnegative. Since $a(t) \leqslant 0$ we can use the properties of k to bound $|x(t_0)|$ by some number $\bar{M}$. We also have

$$\alpha \geqslant \bar{l}(x(t_0)) + \int_T [a(t)\, x(t) + b(t)\, u(t)]\, dt + \int_T b(t)\, \mu(t)\, d\theta(t).$$

By the above

$$\begin{aligned}
\alpha \geqslant k(\bar{M}) &+ \bar{M} \int_T a(t) \ulcorner \Phi(t)\, \Phi^{-1}(t_0)\, e \urcorner\, dt \\
&+ \int_T a(t)\, \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, c(s)\, ds\, dt \\
&+ \int_T b(t)\, u(t)\, dt + \int_T b(t)\, \mu(t)\, d\theta(t).
\end{aligned}$$

Then for each i

$$\bar{\bar{M}} \equiv \frac{1}{\delta}\left(\alpha - k(\bar{M}) - \bar{M}\int_T a(t)\ulcorner \Phi(t)\,\Phi^{-1}(t_0)\,e\urcorner\,dt\right.$$

$$\left. + \int_T a(t)\,\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,c(s)\,ds\,dt\right)$$

$$\geqslant \int_T u_i(t)\,dt + \int \mu_i(t)\,d(t).$$

Since the components of B are L_1^∞ functions we can use their bounds and the bound on $|x(t_0)|$ to bound $\int_T |dx(t)|$ and hence $\|x\|_V$ Q.E.D.

Theorem 7. *Let* (Q0)–(Q4) *be satisfied. Then an optimal solution exists for* (EP) *and the optimal state vector x is equivalent to an $\tilde{x}$ belonging to $\mathscr{A}$. Moreover, this $\tilde{x}$ and its associated control form an optimal solution for* (P).

Proof. The assumptions imply that the various conditions of the previous propositions are met and so we can apply the results of [7] and Theorem 5 to validate the above assertions if we can show that a feasible solution exists.

Let $\bar{x}(t_0)$ be feasible. Then at every point t choose the control $\bar{u}(t)$ by

$$\bar{u}(t) = \max\{0, -g(t)[d(t) + C(t)\,x(t)]\}.$$

This will produce a state vector

$$\bar{x}(t) = \bar{x}(t_0) + \int_{t_0}^{t} [A(s)\,\bar{x}(s) + B(s)\,\bar{u}(s) + c(s)]\,ds$$

and the pair $(\bar{x}, \bar{u})$ will be feasible and have finite value for both (P) and (EP). Q.E.D.

THEOREM 8. *Let* (Q0)–(Q4) *hold. Then an optimal solution exists for* (ED) *and the optimal state vector p is equivalent to a $\tilde{p}$ belonging to $\mathscr{A}$. Moreover, this $\tilde{p}$ and its associated control form an optimal solution for* (D).

Proof. As for the previous theorem, we will use the results of [7] to prove the above. The Hamiltonian for (D) and (ED) is

$$\begin{aligned} \tilde{H}(t, p, x) &= -\infty & &\text{if } p \notin \bar{P}(t) \\ &= -H(t, x, p) & &\text{otherwise} \end{aligned}$$

so the previous propositions also imply (A2)–(A5) are satisfied for (ED). Since L is a normal integrand, its dual, the Lagrangian for (ED) is also a normal integrand, and using the linearity of the problem this implies that the results of Theorem 1A of [7] and a generalization of the Equivalence Theorem of [12] hold. Also, Theorem 4 holds.

What remains is to show that the level sets of Ψ are bounded in the $\|\cdot\|_V$ norm and that a feasible solution exists. By the properties of $\bar{l}$, we see that

$$p(t_1) = 0.$$

We also have

$$0 \leqslant w_i(t) \leqslant |b(t) - p(t)\, B(t)|\, g(t).$$

Then using

$$\dot{p}(t) = -p(t)\, A(t) - w(t)\, C(t) + a(t)$$

we obtain the appropriate bounds.

Because $A(t) \geqslant 0$, $C(t) \geqslant 0$ and $a(t) \leqslant 0$ we see that $p(t) \geqslant 0$ for all t, so that

$$0 \leqslant b(t) - p(t)\, B(t)$$

and zero is feasible for w. The properties of $\bar{l}$ ensure that $p(t_0)$ is unconstrained so the dual state $\bar{p}$ generated by using the zero control and terminating at the origin is a feasible solution with finite cost. Q.E.D.

4. Continuous Linear Programming

Continuous linear programming is a class of problems that has been given much attention in the literature [2, 5, 13]. It consists of the following problem and its dual.

$$\text{(LP)} \quad \text{maximize} \int_T -b(t)\, u(t)\, dt$$

$$\text{subject to } -D(t)\, u(t) \leqslant d(t) + \int_{t_0}^{t} K(t, s)\, u(s)\, ds,$$

$$u(t) \geqslant 0.$$

$$\text{(LD)} \quad \text{minimize} \int_T w(t)\, d(t)\, dt$$

$$\text{subject to } -w(t)\, D(t) \geqslant -b(t) + \int_{t}^{t_1} w(s)\, K(s, t)\, ds,$$

$$w(t) \geqslant 0.$$

Comparing these problems to (P) and (D) of the previous section, we see that by considering K such that

$$K(t, s) = C(t)\, B(s)$$

and restricting (P) and (D) as follows

$$\begin{aligned} l(x_0, x_1) &= 0 \qquad && \text{if } x_0 = 0 \\ &= +\infty && \text{otherwise} \end{aligned}$$

$\forall t \in T$, $a(t)$, $c(t)$ and $A(t)$ all have components that are zero, and

$$\begin{aligned} m(p_0, p_1) &= 0 \qquad && \text{if } p_1 = 0 \\ &= +\infty && \text{otherwise} \end{aligned}$$

then we can treat continuous linear programming problems as a subclass of linear control problems.

Then the results for (P) and (D) also hold true for (LP) and (LD). In particular, the assumptions in cases (i) and (ii) of Theorem 5 become equivalent, and, combined with (32), they lie at the heart of the hypotheses required for the existence of optimal solutions to (LP) and (LD). One also finds the dual conditions to the above, that is, those in cases (iii) and (iv) of Theorem 6 as well as (31).

Comparing Theorems 7 and 8 with the results obtained in the continuous programming literature for problems of the form (LP) and (LD) we see some slight differences. Their results do not require that b is bounded away from zero and they do not assume the continuity of b and B. However, their assumptions are more restrictive on the behaviour of D.

5. Nonlinear Control Problems

Continuous linear programming problems have been generalized in different ways to nonlinear problems. See, for example, [1, 3, 4]. As was the case for continuous linear programming, results have been obtained that ensure that an optimal control vector u will exist and will belong to L^1_m, however, the nonlinearity will mean that the primal problem and any associated dual problem can be of very different form. In this section, therefore, we will not attempt to develop corresponding results for a dual problem but will be satisfied with the investigation of the (primal) problem.

The problem we will consider is

$$\underset{\substack{x \in \mathscr{B} \\ v \in \mathscr{B}_m}}{\text{minimize}}\ l(x(t_0), x(t_1)) + \int_T f_0(t, x(t), u(t))\, dt$$

$$+ \int_T \hat{r}_0(t, x(t), \mu_s(t))\, d\theta_s(t) + \sum_{t \in T} \hat{q}_0(t, x(t^-), x(t^+))$$

where $\hat{r}_0(t, x, \cdot)$ is the recession functon for $f_0(t, x, \cdot)$ and

$$\hat{q}_0(t, a, b) = \inf \Big\{ \int_0^1 \hat{r}_0(t, y(s), z(s))\, ds \colon y(0) = a,\ y(1) = b,$$

$$y \in \mathscr{A}[0, 1],\ z \in L_m^1[0, 1] \text{ with } \dot{y}(s) = B(t)\, z(s),$$

$$r_g(t, x(s)) \leqslant 0 \text{ and } z(s) \geqslant 0 \Big\}$$

subject to

$$\begin{aligned} dx(t) &= [f(t, x(t)) + B(t)\, u(t)]\, dt + B(t)\, \mu(t)\, d\theta(t) \\ dv(t) &= u(t)\, dt + \mu(t)\, d\theta(t) \\ &= u(t)\, dt + \mu_s(t)\, d\theta_s(t) + \mu_a(t)\, d\theta_a(t) \\ &\quad g(t, x(t), u(t)) \leqslant 0 \\ &\quad r_g(t, \mu(t)) \leqslant 0 \\ &\quad u(t) \geqslant 0 \\ &\quad \mu(t) \geqslant 0 \end{aligned}$$

where $f_0(t, x, \cdot)$ and $g(t, \cdot, \cdot)$ are convex and r_g is the recession function for g (see [7] for the notation used here).

THEOREM 9. *Let (x^*, v^*) be optimal. Assume that $\forall t \in T$, $f(t, \cdot)$ is nondecreasing and for all feasible u, $f_0(t, \cdot, u)$ and $g(t, \cdot, u)$ are nonincreasing. Also assume that $\forall t \in T$,*

$$r_g(t, \mu) \leqslant 0, \qquad \mu \geqslant 0$$

implies

$$B(t)\, \mu \leqslant 0 \qquad \text{and} \qquad \hat{r}_0(t, \cdot, \mu) \geqslant 0.$$

Let $l(x_0, x_1) = \bar{l}(x_0)$. Then there exists an optimal pair $(\hat{x}, \hat{v})$ with $\hat{x} \in \mathscr{A}$ and $d\hat{v}(t) = u^*(t)\, dt$.

Proof. Let $\hat{x}(t_0) = x^*(t_0)$ so that $\bar{l}(\hat{x}(t_0)) = \bar{l}(x^*(t_0))$. Also.

$$\begin{aligned} dx^*(t) &= [f(t, x^*(t)) + B(t)\, u^*(t)]\, dt + B(t)\, \mu^*(t)\, d\theta^*(t) \\ &\leqslant [f(t, x^*(t)) + B(t)\, u^*(t)]\, dt \\ &= d\hat{x}(t) \end{aligned}$$

when $t = t_0$. Since $f(t, \cdot)$ is nondecreasing we obtain

$$x^*(t) \leqslant \hat{x}(t).$$

Then

$$\begin{aligned} \Psi(x^*, v^*) &\geqslant \bar{l}(\hat{x}(t_0)) + \int_T f_0(t, \hat{x}(t), u^*(t))\, dt \\ &= \Psi(\hat{x}, \hat{v}). \end{aligned}$$

We still need to show the feasibility of $(\hat{x}, \hat{v})$. This requires

$$g(t, \hat{x}(t), u^*(t)) \leqslant 0.$$

But $g(t, x^*(t), u^*(t)) \leqslant 0$, $x^*(t) \leqslant \hat{x}(t)$ and $g(t, \cdot, u)$ is nonincreasing so the above is true. Q.E.D.

Remark. This theorem only imitates the first case of Theorem 5. By following the above procedure, we can obtain similar results that match the remaining three cases.

If we also place conditions on the above problem that ensure the hypotheses of Theorem 4 of [7] are met, then if the problem is feasible, it will have an optimal solution where the state vector belongs to $\mathscr{A}$.

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